Obstructions to deforming space curves and non-reduced components of the Hilbert scheme

Hirokazu Nasu

Tokai University

March 5, 2013, @HIOA
§1 Introduction

§2 Infinitesimal analysis of the Hilbert scheme
§3 Application to Kleppe’s conjecture
§4 Obstruction to deforming curves on a quartic surface

§1.1 Conventions and Notation
§1.2 Mumford’s example
Hilbert scheme

$V$: a projective variety over $k = \overline{k}$. $\text{char } k = 0$

$H$: an ample divisor on $V$.

**Notation**

\[
\begin{align*}
\text{Hilb} V & \quad = \text{the (full) Hilbert scheme of } V \\
\bigcup \text{ open} \\
\text{Hilb}^{sc} V : & \quad = \{\text{smooth connected curves } C \subset V\} \\
\text{closed } \bigcup \text{ open} \\
\text{Hilb}^{sc}_{d,g} V : & \quad = \{\text{curves of degree degree } d \text{ and genus } g\} \\
(d := (C \cdot H)_V)
\end{align*}
\]
1 Introduction

2 Infinitesimal analysis of the Hilbert scheme

3 Application to Kleppe’s conjecture

4 Obstruction to deforming curves on a quartic surface

1.1 Conventions and Notation

1.2 Mumford’s example

Hilbert scheme of space curves

\[ V = \mathbb{P}^3: \text{the projective 3-space over } k \]

\[ C \subset \mathbb{P}^3: \text{a closed subscheme of } \dim \ = \ 1 \]

\[ d(C): \text{degree of } C \ (= \ #(C \cap H)) \]

\[ g(C): \text{genus of } C \ (\text{as a cpt. Riemann surf.}) \]

We study the Hilbert scheme of space curves:

\[ H^{S}_{d, g} := \text{Hilb}^{sc}_{d, g} \mathbb{P}^3 \]

\[ = \left\{ C \subset \mathbb{P}^3 \mid \text{smooth and connected} \right\} \]

\[ d(C) = d \text{ and } g(C) = g \]
Some reasons are:

- For every smooth curve $C$, there exists a curve $C' \subset \mathbb{P}^3$ s.t. $C' \simeq C$.
- $\text{Hilb}^{sc} \mathbb{P}^3 = \bigsqcup_{d,g} H^S_{d,g}$
- More recently, the classification of the space curves has been applied to the study of bir. automorphism $\Phi : \mathbb{P}^3 \to \mathbb{P}^3$

(for the construction of Sarkisov links [Blanc-Lamy, 2012]).
Some basic facts

- If $g \leq d - 3$, then $H^S_{d,g}$ is irreducible [Ein,’86] and $H^S_{d,g}$ is generically smooth of expected dimension $4d$.
- In general, $H^S_{d,g}$ can become reducible, e.g.
  \[ H^S_{9,10} = W^{(36)}_1 \sqcup W^{(36)}_2 \]  [Noether].
- The Hilbert scheme of arith. Cohen-Macaulay (ACM, for short) curves are smooth [Ellingsrud, ’75].
  \[ C \subset \mathbb{P}^3: \text{ACM} \iff H^1(\mathbb{P}^3, \mathcal{I}_C(l)) = 0 \text{ for all } l \in \mathbb{Z} \]
- $H^S_{d,g}$ can have many generically non-reduced irreducible components, e.g. [Mumford’62], [Kleppe’87], [Ellia’87], [Gruson-Peskine’82], etc.
Infinitesimal property of the Hilbert scheme

$V$: a projective variety over $k$
$C \subset V$: a subvariety of $V$
$I_C$: the ideal sheaf defining $C$ in $V$
$N_{C/V}$: the normal sheaf of $C$ in $V$

Fact (Tangent space and Obstruction group)

1. The tangent space of $\text{Hilb } V$ at $[C]$ is isomorphic to $\text{Hom}(I_C, O_C) \cong H^0(C, N_{C/V})$
2. Every obstruction $\text{ob}$ to deforming $C$ in $V$ is contained in the group $\text{Ext}^1(I_C, O_C)$. If $C$ is a locally complete intersection in $V$, then $\text{ob}$ is contained in $H^1(C, N_{C/V})$
$W \subset \text{Hilb } V$: an irreducible closed subset of $\text{Hilb } V$.

$[C] \in W$: a closed point of $W$

$C_\eta \in W$: the generic point of $W$

**Definition**

- We say $C$ is **unobstructed** (resp. **obstructed**) (in $V$) if $\text{Hilb } V$ is **nonsingular** (resp. **singular**) at $[C]$.

- We say $\text{Hilb } V$ is **generically smooth** (resp. **generically non-reduced**) along $W$ if $\text{Hilb } V$ is **nonsingular** (resp. **singular**) at $C_\eta$. 
Mumford’s example (a pathology)

\[ S \subset \mathbb{P}^3: \text{a smooth cubic surface} \quad (\simeq \text{Blow}_{6 \text{pts}} \mathbb{P}^2) \]
\[ h = S \cap \mathbb{P}^2: \text{a hyperplane section} \]
\[ E: \text{a line on } S \]

There exists a smooth connected curve

\[ C \in \mathcal{H}_{S}^{14,24} \subset \text{Hilb}^{sc} \mathbb{P}^3 \]

of degree 14 and genus 24.

Then \( C \) is parametrized by a locally closed subset

\[ W = W^{(56)} \subset \mathcal{H}_{14,24}^S \subset \text{Hilb}^{sc} \mathbb{P}^3 \]

of the Hilbert scheme.
The locally closed subset $W^{(56)}$ fits into the diagram

$$\left\{ C \subset \mathbb{P}^3 \mid C \subset \mathbb{P}^3 \text{ (smooth cubic)} \right\} =: W^{(56)} \subset H^S_{14,24}$$

$$\downarrow \mathbb{P}^{39} \text{-bundle}$$

$$=: U \subset \text{open } |O_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{19},$$

where we have $\dim |O_S(C)| = 39$ and $h^0(N_{C/\mathbb{P}^3}) = 57$. 

Hirokazu Nasu
Obstructions to deforming space curves and...
$H^0(N_C/\mathbb{P}^3)$ = the tangent space of $\text{Hilb}^{sc} \mathbb{P}^3$ at $[C]$. We have the following inequalities:

$$56 = \dim W \leq \dim_{[C]} \text{Hilb}^{sc} \mathbb{P}^3 \leq h^0(N_C/\mathbb{P}^3) = 57.$$ 

Thus we have a dichotomy between (A) and (B):

- **A** $W$ is an irreducible component of $(\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}}$.
  $\text{Hilb}^{sc} \mathbb{P}^3$ is generically non-reduced along $W$.

- **B** There exists an irreducible component $W' \not\supseteq W$.
  $\text{Hilb}^{sc} \mathbb{P}^3$ is generically smooth along $W$.

Which? $\sim$ The answer is (A). (It suffices to prove $\text{Hilb}^{sc} \mathbb{P}^3$ is singular at the generic point $[C]$ of $W$. We will see later in §2)
Later many non-reduced components of $\text{Hilb}^{sc} \mathbb{P}^3$ were found by Kleppe['85], Ellia['87], Gruson-Peskine['82], Fløystad['93] and Nasu['05]. Moreover, to the question "How bad can the deformation space of an object be?", Vakil['06] has answered that

Law (Murphy’s law in algebraic geometry)

Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.
A naive question

Nowadays non-reduced components of Hilbert schemes are not seldom. However,

**Question**

What is/are the most important reason(s) (if any) for their existence?

Our answer is the following: (at least in Mumford’s example,) a $(-1)$-curve $E$ (i.e. $E \cong \mathbb{P}^1$, $E^2 = -1$) on the (cubic) surface is the most important.
A generalization of Mumford’s ex.

**Theorem (Mukai-Nasu’09)**

\(V\): a smooth projective 3-fold. Suppose that

1. there exists a curve \(E \simeq \mathbb{P}^1 \subset V\) s.t. \(N_{E/V}\) is generated by global sections,
2. there exists a smooth surface \(S\) s.t. \(E \subset S \subset V\), \((E^2)_S = -1\) and \(H^1(N_{S/V}) = p_g(S) = 0\).

Then the Hilbert scheme \(\text{Hilb}^{sc} V\) has infinitely many generically non-reduced components.

In Mumford’s ex., \(V = \mathbb{P}^3\), \(S\): a smooth cubic, \(E\): a line.
Examples

We have many ex. of generically non-reduced components of $\text{Hilb}^{sc} V$ for uniruled 3-folds $V$.

Ex.

1. Let $V$ be a Fano 3-fold and let $-K_V = H + H'$, where $H, H'$: ample. $\exists S \in |H|$ (smooth).
   If $S \neq \mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then there exists a $(-1)\mathbb{P}^1 E$ on $S$.

2. Let $V \to F$ be a $\mathbb{P}^1$-bundle over a smooth surface $F$ with $p_g(F) = 0$. Let $S_1$ be a section of $\pi$ and $A$ a sufficiently ample divisor on $F$. Then there exists a smooth surface $S \in |S_1 + \pi^* A|$. Take a fiber $E$ of $S \to F$. 
§2 Infinitesimal analysis of the Hilbert scheme
In the analysis of Mumford’s ex., we develop some techniques to computing the obstruction to deforming a curve on a uniruled 3-fold ("obstructedness criterion").

**Setting:**
- \( V \): a uniruled 3-fold
- \( S \): a surface
- \( E: (−1)\mathbb{P}^1 \) on \( S \)
- \( C \): a curve on \( S \)
  with \( C \subset S \subset V \)

**Obstruction Criterion**

Non-reduced components of \( \text{Hilb}^{sc} V \)
Obstructions and Cup products

$\tilde{C} \subset V \times \text{Spec } k[t]/(t^2)$: a first order (infinitesimal) deformation of $C$ in $V$ (i.e., a tangent vector of $\text{Hilb } V$ at $[C]$)

Define the cup product $\text{ob}(\alpha)$ by

$$\text{ob}(\alpha) := \alpha \cup e \cup \alpha \in \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_C),$$

where $e \in \text{Ext}^1(\mathcal{O}_C, \mathcal{I}_C)$ is the ext. class of an exact sequence $0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0$. 
Fact

A first order deformation \( \tilde{C} \) lifts to a deformation over \( \text{Spec } k[t]/(t^3) \) if and only if \( \text{ob}(\alpha) = 0 \).

Remark

1. If \( \text{ob}(\alpha) \neq 0 \), then \( \text{Hilb } V \) is singular at \([C]\).
2. If \( C \) is a loc. complete intersection in \( V \), then \( \text{ob}(\alpha) \) is contained in the small group \( H^1(C, N_{C/V}) \) \( (\subset \text{Ext}^1(I_C, \mathcal{O}_C)) \).
Let $C \subset S \subset V$ be a flag of a curve, a surface and a 3-fold (all smooth), and let $\pi_{C/S} : N_{C/V} \to N_{S/V}|_C$ be the natural projection.

**Definition**

Define the *exterior component* of $\alpha$ and $\text{ob}(\alpha)$ by

\[
\begin{align*}
\pi_S(\alpha) & := H^0(\pi_{C/S})(\alpha) \\
\text{ob}_S(\alpha) & := H^1(\pi_{C/S})(\text{ob}(\alpha)),
\end{align*}
\]

respectively.
Let $E \subset S \subset V$ be a flag of a curve, a surface and a 3-fold.

**Definition**

A rational section $v$ of $N_{S/V}$ admitting a pole along $E$, i.e.

$$v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V}),$$

is called an **infinitesimal deformation with a pole**.

**Remark (an interpretation)**

Every inf. def. with a pole induces a 1st ord. def. of the open surface $S^o = S \setminus E$ in $V^o = V \setminus E$ by the map

$$H^0(N_{S/V}(E)) \leftrightarrow H^0(N_{S^o/V^o})$$
Now we are ready to give a sufficient condition for a first order infinitesimal deformation of $\tilde{C}$ ($\subset V \times \text{Spec } k[t]/(t^2)$) of $C$ in $V$ to a second order deformation $\tilde{\tilde{C}}$ ($\subset V \times \text{Spec } k[t]/(t^3)$).
We consider \( \alpha \in H^0(N_{C/V}) \) satisfying the following condition (\( \star \)): the ext. comp. \( \pi_S(\alpha) \) of \( \alpha \) lifts to an inf. def. with a pole along \( E \), say \( v \), and its restriction \( v\big|_E \) to \( E \) does not belong to the image of the map \( \pi_{E/S}(E) := \pi_{E/S} \otimes O_S(E) \).

\[
\begin{align*}
H^0(N_{C/V}) & \ni \alpha \\
\downarrow \pi_{C/S} & \|
\downarrow \\
H^0(N_{S/V}|_C) & \ni \pi_S(\alpha) \\
\overset{ (=v|_C) }{\cdots} & \leftarrow v \overset{ \text{res.} }{\leftrightarrow} v\big|_E \ni \left. H^0(N_{S/V}(E)) \right|_E \\
\cap & \quad \ni \\
\left. H^0(N_{S/V}(E)) \right|_C & \leftarrow \quad H^0(N_{S/V}(E))
\end{align*}
\]
Theorem (Mukai-Nasu’09)

Let \( C, E \subset S \subset V \) be as above. Suppose that \( E^2 < 0 \) on \( S \), and let \( \alpha \in H^0(N_{C/V}) \) satisfy (\( \star \)). If moreover,

1. Let \( \Delta := C + K_V|_S - 2E \) on \( S \). Then

\[
(\Delta \cdot E)_S = 2(-E^2 + g(E) - 1)
\]  

(2.1)

2. the res. map \( H^0(S, \Delta) \to H^0(E, \Delta|_E) \) is surjective, then we have \( \text{ob}_S(\alpha) \neq 0 \).

Remark

If \( E \) is a \((-1)\)-\( \mathbb{P}^1 \) on \( S \), then the RHS of (2.1) is equal to 0.
(Mumford’s ex. $V = \mathbb{P}^3$)

Every general member $C \subset \mathbb{P}^3$ of Mumford’s ex. $W^{(56)} \subset \text{Hilb}^{sc} \mathbb{P}^3$ is contained in a smooth cubic surface $S$ and $C \sim 4h + 2E$ on $S$ ($E$: a line, $h$: a hyp. sect.).

Let $t_W$ denote the tangent space of $W$ at $[C]$ ($\dim t_W = \dim W = 56$).

Then there exists a first order deformation

$$\tilde{C} \leftrightarrow \alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W.$$ 

of $C$ in $\mathbb{P}^3$.

**Claim**

$\text{ob}(\alpha) \neq 0$. 

Hirokazu Nasu  
Obstructions to deforming space curves and ...
Proof.

Since $H^1(N_{S/\mathbb{P}^3}(E - C)) = 0$, the ext. comp. $\pi_{C/S}(\alpha) \in H^0(N_{S/\mathbb{P}^3}|_C)$ of $\alpha$ has a lifts to a rational section $v \in H^0(N_{S/\mathbb{P}^3}(E))$ on $S$ (an inf. def. with a pole). By the key lemma below, the restriction $v|_E$ to $E$ is not contained $\text{im} \pi_{E/S}(E)$. Since $C \sim 4h + 2E = -K_{\mathbb{P}^3}|_S + 2E$, the divisor $\Delta$ is zero. Thus the condition (1) and (2) are both satisfied.

Lemma (Key Lemma)

Since $C$ is general, the finite scheme $Z := C \cap E$ of length 2 is not cut out by any conic in $|h - E| \cong \mathbb{P}^1$ on $S$. 

\[ \square \]
§3 Application to Kleppe’s conjecture
§3.1 Minimal degree and Maximal subsets

§3.2 Kleppe’s conjecture

§3.3 Main Result

**Minimal degree** \( s(W) \) for \( W \subset \mathcal{H}_{d,g}^S \)

\[
\mathcal{H}_{d,g}^S \subset \mathcal{H}_{d,g}^{sc} \mathbb{P}^3: \text{the subsch. of curves of deg. } d \text{ and gen. } g
\]

\( W \subset \mathcal{H}_{d,g}^S: \text{an irreducible closed subset} \)

\( C \in W: \text{a general member of } W \)

**Definition (minimal degree of } W)\)

\[
s(W) := \min \left\{ s \in \mathbb{N} \mid H^0(\mathbb{P}^3, \mathcal{I}_C(s)) \neq 0 \right\}
\]

\[= \text{the minimal degree of a surface } S \supset C\]
$s$-maximal subsets of $H^S_{d,g}$

**Definition (Kleppe’87)**

$W \subset H^S_{d,g}$ is called $s(W)$-maximal if it is maximal w.r.t $s(W)$.

$W$: $s$-maximal $\Rightarrow s(V) > s$ for any closed irreducible subset $V \nsubseteq W$.

**Ex. (Mumford’s ex.)**

$$W = \left\{ C \subset \mathbb{P}^3 \mid C \subset \exists S \text{ (sm. cubic)} \text{ and } C \sim 4h + 2E \right\}^-$$

is a $3$-maximal subset of $H^S_{14,24}$. 
First properties of $s$-maximal subsets

In what follows, we assume that

1. $W \subset H^S_{d,g}$: a $s$-maximal subset
2. a general member $C \subset S$, where $S$: a smooth surface of deg $s$.

**Proposition**

Suppose that $s \leq 4$ and $d > s^2$. Then

1. If $C$ is not a c.i. when $s = 4$, then 
   \[
   \dim W = (4 - s)d + g + \binom{s+3}{s} - 2
   \]
2. If $H^1(\mathbb{P}^3, I_C(s)) = 0$ and if $C$ is not a c.i. when $s = 4$, then $W$ is a **generically smooth** component of $H^S_{d,g}$. 

Hirokazu Nasu
Obstructions to deforming space curves and ...
Let $s = 3$. If $d > 3^2 = 9$, then $\dim W = d + g + 18$.

**Fact**

Every irreducible component of $H^S_{d,g}$ is of dimension greater than or equal to $4d \ (= \chi(N_C/\mathbb{P}^3))$

Thus if $W$ is a component, then

$$\dim W \geq 4d \iff g \geq 3d - 18.$$
Conjecture (Kleppe’87, a ver. modified by Ellia)

Let $d > 9$, $g \geq 3d - 18$, and let $W \subset H^S_{d,g}$ be a 3-maximal subset. If a general member $C$ of $W$ satisfies

1. $H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0$, and
2. $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$

then $W$ is a gen. non-reduced irred. component of $H^S_{d,g}$. 
Remark

1. The linearly normality assumption \( H^1(I_C(1)) = 0 \) was missing in the original ver. of the conjecture. (pointed out by Ellia[’87] with a counterexample).

2. The tangential dimension \( h^0(N_{C/\mathbb{P}^3}) \) of \( H^S_{d,g} \) at \([C] \) is greater than \( \dim W \) by \( h^1(I_C(3)) \).

3. The subset \( W \) can be described more explicitly in terms of the coordinate \((a; b_1, \ldots, b_6)\) of \( C \) in \( \text{Pic} \ S \cong \mathbb{Z}^7 \)
Known results

In the following cases, Kleppe’s conjecture is known to be true.

1. \( g > 7 + \frac{(d - 2)^2}{8} \) and \( d \geq 18 \) [Kleppe’87]

2. \( d \geq 21 \) and \( g > G(d, 5) \) [Ellia’87]

3. \( h^1(\mathbb{P}^3, I_C(3)) = 1 \) [Nasu’05]

\(^1\) \( G(d, 5) \) denotes the max. genus of curves of degree \( d \), not contained in any quartic surface. \( G(d, 5) \approx d^2/10 \) for \( d \gg 0 \).

\(^2\)proved by computing cup products.
Let $d > 9$ and $g \geq 3d - 18$ and let $W \subset H^S_{d,g}$ and $C$ as in Kleppe’s conjecture.

**Lemma**

The following conditions are equivalent:

1. $C$ is **quadratically normal**, i.e., $H^1(\mathbb{P}^3, I_C(2)) = 0$.
2. $(C \cdot E) \geq 2$ for every line $E$ on $S$
3. Let $C \sim (a; b_1, \ldots, b_6)$ with some basis of $\text{Pic } S \cong \mathbb{Z}^7$. Then $b_i \geq 2$ for all $i = 1, \ldots, 6$.
4. Let $h \in \text{Pic } S$ be the cls. of hyp. sections. The base locus of the complete lin. sys. $|C - 3h|$ contains no double lines $2E_i$, and no triple lines $3E_i$. 
Main Theorem

Theorem (—’09)

Kleppe’s conjecture is true if $C$ is quadratically normal, i.e.,

$$H^1(\mathbb{P}^3, I_C(2)) = 0.$$
How to prove Main Theorem

As we see in the Mumford’s ex., it suffices to prove that \( \text{ob}(\alpha) \neq 0 \) for every

\[
\alpha \in H^0(C, N_{C/P^3}) \setminus t_W,
\]

where \( t_W \) is the tangent space of \( W \) at \([C]\).

Note that the ext. comp. \( \pi_S(\alpha) \) of \( \alpha \) lifts to a rational section

\[
v \in H^0(S, N_{S/P^3}(F)) \setminus H^0(S, N_{S/P^3}),
\]

where \( F \) is the fixed component of the lin. sys. \(|C - 3h|\).

Then we apply the obstructedness criterion for a first order deformation \( \tilde{C} \leftrightarrow \alpha \) of \( C \) in \( P^3 \).
Another progress has been made:

Theorem (Kleppe’12)

Kleppe’s conjecture is true provided that:

1. \( b_6 = 2, b_5 \geq 4, d \geq 21 \) and
   \[(a; b_1, \ldots, b_6) \neq (\lambda + 12, \lambda + 4, 4, \ldots, 4, 2), \forall \lambda \geq 2,\]
2. \( b_6 = 1, b_5 \geq 6, d \geq 35 \) and
   \[(a; b_1, \ldots, b_6) \neq (\lambda + 18, \lambda + 6, 6, \ldots, 6, 1), \forall \lambda \geq 2,\]
3. \( b_6 = 1, b_5 = 5, b_4 \geq 7, d \geq 35 \) and
   \[(a; b_1, \ldots, b_6) \neq (\lambda + 21, \lambda + 7, 7, \ldots, 7, 5, 1), \forall \lambda \geq 2.\]
§4 Obstruction to deforming curves on a quartic surface
Quartic surfaces containing a line

Similarly, we can compute the obstructions to deforming curves on a smooth quartic surface.

Assume that:
- \( S \subset \mathbb{P}^3 \): a smooth quartic surface (a K3 surface),
- \( E \): a line on \( S \),
- \( F (\sim h - E) \): a plane cubic curve cut out by a plane \( H \supset E \),
- \( \text{Pic} \, S \cong \mathbb{Z}E \oplus \mathbb{Z}F \), and the intersection matrix is given by

\[
\begin{pmatrix}
  E^2 & E \cdot F \\
  E \cdot F & F^2 \\
\end{pmatrix}
= 
\begin{pmatrix}
  -2 & 3 \\
  3 & 0 \\
\end{pmatrix}.
\]
Then every curve $C$ on $S$ is expressed in $\text{Pic} S$ by

$$C \sim aE + bF \quad (a, b \geq 0)$$

with $d = a + 3b$ and $g = 3ab - b^2 + 1$.

Let $W$ be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$

Moreover, we see that....
Theorem (Kleppe’12 (with Ottem))

Suppose that $d > 16$ and $4 < a < b$. Then

1. If $3b - 2a \geq 3$, then $h^1(I_C(4)) = 0$. In particular, $W$ is a \textbf{generically smooth} component of $H^S_{d,g}$.

2. If $3b - 2a \leq 2$, then $h^1(I_C(4)) \neq 0$. Moreover, $W$ is a \textbf{generically non-reduced} component of $H^S_{d,g}$.

In fact, we see that

$$h^1(\mathbb{P}^3, I_C(4)) = \begin{cases} 
1 & (3b - 2a = 2) \\
2 & (3b - 2a = 1) \\
4 & (3b - 2a = 0) 
\end{cases}$$
By computing cup products, we have proved the following:

**Theorem**

Let $C$ and $W$ be as in the thm. If

$$3b - 2a = 2 \quad (\Rightarrow h^1(\mathbb{P}^3, I_C(4)) = 1),$$

then there exists a first order deformation $\tilde{C}$ of $C$ in $\mathbb{P}^3$ which does not lift to a deformation over $\text{Spec} \ k[t]/(t^3)$.

However, for the other case (where $3b - 2a = 1, 0$) we have not yet proved the obstructedness of a general $C \in W$. 
Quartic surfaces containing a conic

We have many variations of a smooth quartic surface $S$ containing $E \simeq \mathbb{P}^1$, e.g., the one containing conics $E_1, E_2$.

Assume that:
$S \subset \mathbb{P}^3$: a smooth quartic surface (a K3 surface),
$E_1, E_2$: conics on $S$ such that $h \sim E_1 + E_2$,
$	ext{Pic} \, S \simeq \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$, and the intersection matrix is given by

$$
\begin{pmatrix}
E_1^2 & E_1 \cdot E_2 \\
E_1 \cdot E_2 & E_2^2
\end{pmatrix}
= 
\begin{pmatrix}
-2 & 4 \\
4 & -2
\end{pmatrix}
$$
Then every curve $C$ on $S$ is expressed in $\text{Pic } S$ by

$$C \sim aE + bF \quad (a, b \geq 0)$$

with $d = 2a + 2b$ and $g = 4ab - a^2 - b^2 + 1$. Let $W$ be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$
Theorem (Kleppe’12 (with Ottem))

Suppose that $d > 16$ and $a \neq b > 4$. If

$$b + 4 \leq a \leq 2b - 4,$$

then $h^1(\mathbb{P}^3, I_C(4)) = 0$. In particular, $W$ is a generically smooth component of $H^S_{d,g}$.

Otherwise, we see that

$$h^1(\mathbb{P}^3, I_C(4)) = \begin{cases} 
1 & (2b - a = 3) \\
4 & (2b - a = 2) \\
9 & (2b - a = 1) \\
16 & (2b - a = 0)
\end{cases}$$
By computing cup products, we have proved the following:

**Theorem**

Let $C$ be a general member of $W$, and suppose that

$$2b - a = 3 \quad (\Rightarrow h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = 1).$$

Then there exists a first order deformation $\tilde{C}$ of $C$ in $\mathbb{P}^3$ which does not lift to a deformation over $\text{Spec } k[t]/(t^3)$.

However, for the other case (where $2b - a = 2, 1, 0$) we have not yet proved the obstructedness of a general $C \in W$. 

Hirokazu Nasu

Obstructions to deforming space curves and ...
S. Mukai and H. Nasu,
Obstructions to deforming curves on a 3-fold I: A generalization of Mumford’s example and an application to Hom schemes.
*J. Algebraic Geom.*, **18**(2009), 691-709

H. Nasu,
Obstructions to deforming curves on a 3-fold, II: Deformations of degenerate curves on a del Pezzo 3-fold,
*Annales de L’Institut Fourier*, **60**(2010), no.4, 1289-1316.