OBSTRUCTIONS TO DEFORMING SPACE CURVES AND A REMARK TO A CONJECTURE OF KLEPPE

HIROKAZU NASU

ABSTRACT. Let us denote by $H^S_{\mathbb{P}^3}$ the Hilbert scheme of smooth connected curves in the projective 3-space $\mathbb{P}^3$. We study the irreducible components of $H^S_{\mathbb{P}^3}$ whose general member C is contained in a smooth cubic surface. In particular, we classify the all such irreducible components of $H^S_{\mathbb{P}^3}$, under the assumption that C is linearly and quadratically normal, i.e., $H^1(\mathbb{P}^3, \mathcal{I}_C(l)) = 0$ for $l = 1, 2$.

1. INTRODUCTION

In this article, curves embedded into the projective 3-space $\mathbb{P}^3$ are called *space curves*. The problem of classifying space curves is classical and existed in the late 19th century [4],[11]. As Hartshorne wrote in his book [5], nowadays the theoretical aspect of this problem has been well understood by virtue of the existence of the Hilbert scheme. However the more specific task of determining the dimension, the irreducibility, etc. of the closed subscheme parametrizing curves of degree d and genus g has not been accomplished for many d and g yet.

Let us denote by $H^S_{\mathbb{P}^3}$ the Hilbert scheme of smooth connected curves in $\mathbb{P}^3$. Among all space curves, curves contained in a cubic surface have been intensively studied by many authors, e.g. [3],[6],[2],[9]. In particular curves on a smooth cubic surface have been studied in detail with a help of the beautiful structure of this surface. We want to know the answer to the following problem.

Problem 1.1. *Classify the irreducible components of the Hilbert scheme $H^S_{\mathbb{P}^3}$ whose general member is contained in a smooth cubic surface.*

Problem 1.1 was studied also by Kleppe [6]. In what follows, we denote by $H^S_{d,g}$ the subscheme of $H^S_{\mathbb{P}^3}$ parametrizing curves of degree d and genus g. Let $W \subset H^S_{d,g}$ be a maximal irreducible closed subset of $H^S_{d,g}$ whose general member C is contained in a smooth cubic surface. (See Definition 2.1 for more explicit description of W.) Kleppe gave the following conjecture, which shall be enough for solving Problem 1.1.

Conjecture 1.2 ([6, Conjecture 4])**.** *Let $d > 9$ and $g \geq 3d - 18$ be two integers and let $W \subset H^S_{d,g}$ and C be as above. Then:*

(1) *W is an irreducible component of $(H^S_{d,g})_{\mathrm{red}}$.*
(2) *$H^S_{d,g}$ is generically smooth along W if and only if $H^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 0$.*

If W is an irreducible component of $H^S_{d,g}$, then we have $g \geq 3d - 18$ (cf. Remark 2.2 (3)). Later Ellia [2] pointed out that the conclusion of the conjecture is false if we do not assume that C is linearly normal (i.e. $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$). He suggested restricting the conjecture to linearly normal curves C. Conjecture 1.2 is known to be true in the following cases:

[i] $d \leq 17$ and $g > -1 + (d^2 - 4)/8$ (by Kleppe [6]);
[ii] $d \geq 18$ and $g > 7 + (d-2)^2/8$ (by Kleppe [6]);
[iii] $d \geq 38$ and $g > (d^2 - 4d + 8)/8$ (by Ellia [2]);
[iv] $d \geq 21$, $g > G(d, 5)$ and C is linearly normal (by Ellia [2]);
[v] $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$ (by Nasu [9]).

Here $G(d, 5)$ denotes the maximal genus of curves of degree d not contained in a quartic surface. In this article, by using a technique to compute obstructions, we prove

Theorem 1.3. *Conjecture 1.2 is true if C is linearly and quadratically normal, i.e. $H^1(\mathbb{P}^3, \mathcal{I}_C(l)) = 0$ for $l = 1, 2$.*

Let C be the general curve in Conjecture 1.2. Since $d > 9$, C is contained in a unique cubic surface S. Then C is linearly and quadratically normal if and only if $C \cdot E \geq 2$ for every line E on S. Here $C \cdot E$ denotes the intersection number of C with E in S.

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2. Curves on a smooth cubic and 3-maximal subsets

First we recall some basic facts on curves on a smooth cubic surface. Let S be a smooth cubic surface. Then as is well known, S is isomorphic to a blown-up of $\mathbb{P}^2$ at six points (which do not lie on a conic, and no three of which lie on a line). The six exceptional curves e_i $(1 \leq i \leq 6)$ together with the pullback l of a line in $\mathbb{P}^2$ form a $\mathbb{Z}$-free basis of the Picard group $\operatorname{Pic} S \simeq \mathbb{Z}^7$ of S. Given a divisor D on S, we obtain a 7-tuple $(a; b_1, \ldots, b_6)$ of integers as the coordinates of the divisor class $D = al - \sum_{i=1}^6 b_i e_i$. Then the Weyl group $W(\mathbb{E}_6)$ associated to the root system of type $\mathbb{E}_6$ acts on $\operatorname{Pic} S$ as follows. Here $W(\mathbb{E}_6)$ is the subgroup generated by the permutations of e_i $(1 \leq i \leq 6)$ and by the additional Cremona element σ given by $\sigma(l) = 2l - e_1 - e_2 - e_3$, $\sigma(e_1) = l - e_2 - e_3$, $\sigma(e_2) = l - e_1 - e_3$, $\sigma(e_3) = l - e_1 - e_2$ and $\sigma(e_i) = e_i$ for $i \notin \{1, 2, 3\}$. Every element of $W(\mathbb{E}_6)$ induces a base change of $\operatorname{Pic} S$. By virtue of the Weyl group and this base change, given a divisor D on S, there exists a suitable blow-up $S \to \mathbb{P}^2$ such that we have

$$b_1 \geq \cdots \geq b_6 \quad \text{and} \quad a \geq b_1 + b_2 + b_3. \tag{2.1}$$

When (2.1) holds, we say that the basis $\{l, e_1, \dots, e_6\}$ of $\operatorname{Pic} S$ is *standard* for D. For the standard basis, the complete linear system $|D|$ on S contains a smooth connected curve C of degree > 2 if and only if $a > b_1$ and $b_6 \geq 0$. The degree d and genus g of C is computed as

$$d = 3a - \sum_{i=1}^{6} b_i \quad \text{and} \quad g = \binom{a-1}{2} - \sum_{i=1}^{6} \binom{b_i}{2}. \tag{2.2}$$

Let (d, g) be a pair of integers with $d > 2$ and let $(a; b_1, \dots, b_6)$ be a 7-tuple of integers satisfying (2.1), (2.2), $a > b_1$ and $b_6 \geq 0$.

Definition 2.1. We define a closed subset $W_{(a;b_1,\dots,b_6)}$ of $H^S_{d,g}$ by

$$W_{(a;b_1,\dots,b_6)} := \left\{ C \in H^S_{d,g} \,\middle|\, \begin{array}{c} C \text{ is contained in a smooth cubic surface } S \\ \text{and } C \sim al - \sum_{i=1}^{6} b_i e_i \text{ on } S \end{array} \right\}^{-}$$

and call $W_{(a;b_1,\dots,b_6)}$ a 3-*maximal subset* of $H^S_{d,g}$.

Here $\{\}^{-}$ denotes the closure of $\{\}$ in $H^S_{d,g}$.

Remark 2.2. Suppose that $d > 9$ and let $W = W_{(a;b_1,\dots,b_6)}$ be a 3-maximal subset of $H^S_{d,g}$. Then by construction we know the following:

(1) W is birationally equivalent to $\mathbb{P}^{d+g-1}$-bundle over $|\mathcal{O}_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19}$, where $d+g-1 = \dim |\mathcal{O}_S(C)|$. In particular W is irreducible and of dimension $d + g + 18$.
(2) W is maximal among all irreducible closed subsets V whose general member is contained in a smooth cubic surface. (If there exists an irreducible closed subset $W' \supsetneqq W$ of $H^S_{d,g}$, then a general member C' of W' is not contained in a cubic surface, but a surface of degree greater than three.)
(3) It is known that every irreducible component of $H^S_{d,g}$ is of dimension at least $4d$. Thus if W is an irreducible component of $H^S_{d,g}$, then we have $g \geq 3d - 18$ by (1).

In what follows, we assume that $d > 9$ and $g \geq 3d - 18$. Let $W \subset H^S_{d,g}$ be a 3-maximal subset and C a general member of W. Let us denote the normal bundle of C in $\mathbb{P}^3$ by $N_{C/\mathbb{P}^3}$. As is well known, the cohomology group $H^0(C, N_{C/\mathbb{P}^3})$ represents the tangent space of the Hilbert scheme $H^S_{d,g}$ at the point $[C]$ corresponding to C. We have a natural inequality

$$\dim W \leq \dim_{[C]} H^S_{d,g} \leq h^0(C, N_{C/\mathbb{P}^3}). \tag{2.3}$$

of dimensions. Then we have the following.

Lemma 2.3. (1) *$H^S_{d,g}$ is nonsingular at $[C]$ if and only if* $\dim_{[C]} H^S_{d,g} = h^0(C, N_{C/\mathbb{P}^3})$.

(2) *W is an irreducible component of $(H^S_{d,g})_{\mathrm{red}}$ if and only if* $\dim W = \dim_{[C]} H^S_{d,g}$.

Furthermore an easy computation shows that

$$h^0(C, N_{C/\mathbb{P}^3}) - \dim W = h^1(\mathbb{P}^3, \mathcal{I}_C(3)). \tag{2.4}$$

Thus if $H^1(\mathcal{I}_C(3)) = 0$, then we have

$$\dim W = \dim_{[C]} H^S_{d,g} = h^0(C, N_{C/\mathbb{P}^3})$$

and Conjecture 1.2 is true. Otherwise the conjecture is not trivial.

We next compute the dimension of the cohomology group $H^1(\mathbb{P}^3, \mathcal{I}_C(3))$ in terms of the coordinate $(a; b_1, \dots, b_6)$ of C in $\operatorname{Pic} S \simeq \mathbb{Z}^{\oplus 7}$. The dimension is closely related to the geometry of lines on the smooth cubic surface containing C. First we recall the celebrated example of Mumford of a generically non-reduced component of $H^S_{14,24}$.

Example 2.4 (Mumford [8])**.** Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, h the class of hyperplane section of S, E a line on S. Then a general member C of the linear system $|4h + 2E|$ is a smooth connected curve of degree 14 and genus 24. Since the coordinate of h in $\operatorname{Pic} S \simeq \mathbb{Z}^{\oplus 7}$ is equal to $(3; 1, 1, 1, 1, 1, 1)$, that of C is equal to $(12; 4, 4, 4, 4, 4, 2)$. The dimension of the 3-maximal subset $W_{(12;4,4,4,4,4,2)}$ is equal to $d+g+18 = 14+24+18 = 56$ by Remark 2.2 (2), while we compute that $h^0(N_{C/\mathbb{P}^3}) = h^0(N_{C/S}) + h^0(N_{S/\mathbb{P}^3}\big|_C) = 37 + 20 = 57$ by using the exact sequence $0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_{S/\mathbb{P}^3}\big|_C \to 0$. Thus we conclude that $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$. Consider the complete linear system $\Lambda_3 := |C - 3h|$ on S. Then we have

$$\Lambda_3 := |4h + 2E - 3h| = |h + 2E| = |h + E| + E,$$

where $|h + E|$ is base point free and E is the base component of Λ_3. It is clear that $h^0(\mathcal{O}_E) = h^1(\mathcal{I}_C(3)) = 1$.

More generally, if the linear system $\Lambda_3 := |C - 3h|$ is free, then we have $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 0$. Otherwise, Λ_3 has a nonzero fixed part F, which becomes a sum of disjoint (but possibly non-reduced) lines on S, and we have $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = h^0(F, \mathcal{O}_F)$. More precisely the following lemma holds.

Lemma 2.5. *Let F be the fixed part of Λ_3. Then we have*

$$F = \sum_{\substack{\text{lines } E \text{ on } S \text{ s.t.} \\ (C - 3h \cdot E) < 0}} -(C - 3h \cdot E)\, E = \sum_{\substack{\text{lines } E \text{ on } S \text{ s.t.} \\ (C \cdot E) < 3}} (3 - (C \cdot E))\, E$$

and $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = h^0(F, \mathcal{O}_F)$.

Let $\{l, e_1, \dots, e_6\}$ be the standard basis of $\operatorname{Pic} S$ for C and $(a; b_1, \dots, b_6)$ the coordinates of C, in other words, $C \sim al - \sum_{i=1}^6 b_i e_i$ on S. In Table 1, we compute $h^1(\mathcal{I}_C(3))$ in the case where the fixed part F is concentrated in the line e_6 on S.

As we can easily guess from the table, for example we have that

Cases	$(a; b_1, \dots, b_6)$	$F = \mathrm{Bs}\,\Lambda_3$	$h^1(\mathbb{P}^3, \mathcal{I}_C(3))$
[0]	$b_6 \geq 3$	$F = \emptyset$	0
[I]	$b_6 = 2, b_5 \geq 3$	e_6	1
[II]	$b_6 = 1, b_5 \geq 3$	$2e_6$	3
[III]	$b_6 = 0, b_5 \geq 3$	$3e_6$	6

TABLE 1. The fixed part F and $h^1(\mathbb{P}^3, \mathcal{I}_C(3))$

- $F = e_5 + e_6 \Longrightarrow h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1 + 1 = 2$,
- $F = e_5 + 2e_6 \Longrightarrow h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1 + 3 = 4$.

We recall that C is said to be *linearly normal* if $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$. In this article, similarly C is said to be *quadratically (resp. cubically) normal* if $H^1(\mathbb{P}^3, \mathcal{I}_C(2)) = 0$ (resp. $H^1(\mathbb{P}^3, \mathcal{I}_C(3))$=0). The following lemma may be useful.

Lemma 2.6. *C is linearly (resp. quadratically, cubically) normal if and only if $b_6 \geq 1$ (resp. $b_6 \geq 2$, $b_6 \geq 3$) under the standard basis of* $\mathrm{Pic}\, S$ *for C.*

This lemma shows that C is quadratically normal if and only if F contains neither of a double line $2E$ nor a triple line $3E$ on S. In other words, F is zero or of the form

$$F = E_1 + E_2 + \cdots + E_m \qquad (m \leq 6), \tag{2.5}$$

where E_i $(1 \leq i \leq m)$ are disjoint reduced lines on S.

3. OBSTRUCTEDNESS CRITERION

In [7] Mukai and Nasu have studied the embedded deformations of a smooth curve C on a smooth projective 3-fold V under the presence of a smooth surface S such that $C \subset S \subset V$. They have given a sufficient condition for a first order infinitesimal deformation of C in V to be obstructed. We roughly explain the main idea of their obstructedness criterion.

Every infinitesimal deformation $\tilde{C}$ of C in V of the first order (i.e. over $\mathrm{Spec}\, k[t]/(t^2)$) determines a global section $\alpha \in H^0(N_{C/V})$ and a cohomology class $\mathrm{ob}(\alpha) \in H^1(N_{C/V})$. Here $\mathrm{ob}(\alpha)$ is called the *obstruction* for $\tilde{C}$ (or α) and $\tilde{C}$ lifts to a deformation over $\mathrm{Spec}\, k[t]/(t^3)$ if and only if $\mathrm{ob}(\alpha) = 0$. Let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be the natural projection. The images of α and $\mathrm{ob}(\alpha)$ by the induced maps $H^i(\pi_{C/S}) : H^i(N_{C/V}) \to H^i(N_{S/V}\big|_C)$ $(i = 0, 1)$ are called the *exterior component* of α and $\mathrm{ob}(\alpha)$, respectively. It is proved in [7, Theorem 2.2] that if there exists a curve E on S such that $(E^2)_S < 0$ (e.g. a (-1)-curve on S) and the exterior component of α lifts to a global section $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$, then the exterior component of $\mathrm{ob}(\alpha)$ is nonzero provided that certain additional conditions on E, C and v are satisfied.

We give a variant of the above theorem, which will be used in our proof of Theorem 1.3. Let $C \subset S \subset V$ be as above and let E_i $(1 \le i \le m)$ be (-1)-curves ($\simeq \mathbb{P}^1$) on S, which are disjoint to each other (i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$). We denote the union $\bigcup_{i=1}^m E_i$ as a scheme or the divisor $\sum_{i=1}^m E_i$ by the same symbol F, whose meaning will be guessed from the context.

Theorem 3.1. *A first order infinitesimal deformation $\tilde{C}$ of C (or $\alpha \in H^0(N_{C/V})$) is obstructed if its exterior component lifts to a global section $v \in H^0(N_{S/V}(F)) \setminus H^0(N_{S/V})$,* i.e., *$\pi_{C/S}(\alpha) = v\big|_C$ in $H^0(N_{S/V}(F)\big|_C)$, and if the following conditions are satisfied:*

(a) *$(\Delta \cdot E_i)_S = 0$ for any $1 \le i \le m$, where we put a divisor $\Delta := C - 2F + K_V\big|_S$ on S,*

(b) *$v\big|_F$ does not belong to the image of*

$$\pi_{F/S}(F) : H^0(F, N_{F/V}(F)) \longrightarrow H^0(F, N_{S/V}(F)\big|_F),$$

(c) *The restriction map $H^0(S, \Delta) \to H^0(F, \Delta\big|_F) \overset{(a)}{\simeq} H^0(F, \mathcal{O}_F) \simeq k^m$ is surjective.*

Figure 1 may be useful for understanding the relation between α and $v\big|_F$ in (b).

$$\begin{array}{ccccccccc}
H^0(N_{C/V}) & \ni & \alpha & & & & & & H^0(N_{F/V}(F)) \\
{\scriptstyle \pi_{C/S}}\big\downarrow & & \big\downarrow & & & & & & {\scriptstyle \pi_{F/S}(F)}\big\downarrow \\
H^0(N_{S/V}\big|_C) & \ni & v\big|_C & \overset{res}{\longleftarrow\!\shortmid} & v & \overset{res}{\shortmid\!\longrightarrow} & v\big|_F & \in & H^0(N_{S/V}(F)\big|_F) \\
\bigcap & & & & \Cap & & & & \\
H^0(N_{S/V}(F)\big|_C) & & \overset{res}{\longleftarrow} & & H^0(N_{S/V}(F)) & & & &
\end{array}$$

Figure 1. Relation between α and $v\big|_F$

4. Sketch of Proof

In this section we sketch the proof of Theorem 1.3. Let W be the 3-maximal subset of $H^S_{d,g}$ as in Theorem 1.3 and C a general member of W, S the smooth cubic surface containing C. If $H^1(\mathcal{I}_C(3)) = 0$ then we are done. Otherwise the tangent space $H^0(N_{C/\mathbb{P}^3})$ of $H^S_{d,g}$ at $[C]$ is greater than $\dim W$ by (2.4). Then by Lemma 2.3, it suffices to show that $\dim_{[C]} H^S_{d,g} = \dim W$ because this implies also that $H^S_{d,g}$ is singular at $[C]$ by the same lemma. Furthermore since C is general in W, $H^S_{d,g}$ is nowhere reduced along W.

We also note that by Lemma 2.5, the linear system $\Lambda_3 = |C - 3h|$ on S has the nonzero fixed part F and we have $h^0(\mathcal{O}_F) = h^1(\mathcal{I}_C(3)) =: m$ $(\neq 0)$. Since C is quadratically normal, F is a sum of disjoint m reduced lines E_i $(1 \le i \le m)$ (cf. (2.5)). Now we apply the obstructedness criterion (Theorem 3.1) to F and check that the dimension of $H^S_{d,g}$ at $[C]$ falls m from $h^0(N_{C/\mathbb{P}^3})$. In fact we can check that for any tangent vector

$\alpha \in H^0(N_{C/\mathbb{P}^3})$ not contained in the tangent space t_W of W, the obstruction $\mathrm{ob}(\alpha)$ is nonzero in $H^1(N_{C/\mathbb{P}^3})$. $\square$

5. Toward a proof of the full conjecture

Conjecture 1.2 (modified by Ellia) is still open. This is because that in proving Theorem 1.3, we have put an extra assumption on the subset W that its general member C is quadratically normal, i.e., $H^1(\mathcal{I}_C(2)) = 0$. According to Ellia's counterexample, if C is not linearly normal, the conclusion of Conjecture 1.2 does not hold. Therefore, in order to prove the full conjecture, we have only to prove it for C, which is not quadratically normal but linearly normal. Then the fixed part F in Lemma 2.5 is of the form

$$F = E_1 + \cdots + E_j + 2E_{j+1} + \cdots + 2E_m,$$

where E_i $(1 \le i \le m)$ are disjoint m lines on the cubic surface $S \supset C$ and $1 \le j \le m$. Therefore the fixed part F contains a non-reduced double line in general, for which it seems hard to prove the full of Conjecture 1.2.

6. Deformations of space curves lying on a surface of low degrees

Another motivation for considering Problem 1.1 comes from the fact that curves C in $\mathbb{P}^3$ lying on a plane (i.e. C is a plane curve), or a quadric surface are easy to classify. For example, all plane curves of degree d in $\mathbb{P}^3$ form the one and only irreducible component of the Hilbert scheme $H^S_{d,(d-1)(d-2)/2}$, which is reduced for all d, and of dimension $(d^2 + 3d + 6)/2$ for $d > 1$. Every irreducible component of $H^S_{\mathbb{P}^3}$ whose general member C is contained in a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ are classified by the degree d and the bidegree (a, b) of C in Q. Given a pair (a, b) of integers satisfying $d = a + b$, $g = (a - 1)(b - 1)/2$ and $a \ge b > 0$, we define an irreducible closed subset of the Hilbert scheme $H^S_{d,g}$ by

$$W_{(a,b)} := \left\{ C \in H^S_{d,g} \;\middle|\; \begin{array}{c} C \text{ is contained in a smooth quadric surface } Q \\ \text{and of bidegree } (a,b) \text{ on } Q \end{array} \right\}^{-}.$$

Then $W_{(a,b)}$ becomes an irreducible component of $H^S_{d,g}$ if and only if $g \ge 2d - 8$. $H^S_{d,g}$ is generically smooth along $W_{(a,b)}$ for all (a, b), and $\dim W_{(a,b)} = g + 2d + 8$ if $d > 4$.

We review a few known results on deformations of space curves lying on a singular cubic surface and conclude this section. Brevik [1] proved in his Ph.D. thesis (1996) that

Proposition 6.1. *Every locally Cohen-Macaulay curve on a rational normal cubic surface is a specialization of curves on a smooth cubic surface.*

Gruson and Peskine showed that there exist many irreducible (and moreover, non-reduced) component of the Hilbert scheme $H^S_{\mathbb{P}^3}$ whose general member is contained in a non-normal cubic surface which is not a cone. The minimal desingularization of this cubic surface is isomorphic to the ruled surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. In fact they proved that

Proposition 6.2 ([3, Proposition B.2]). (1) *The smooth connected curves C in $\mathbb{P}^3$ of degree $d \geq 10$,*

[i] *contained in a non-normal cubic surface S which is not a cone, and*

[ii] *meeting a general ruling line of S at k points,*

form a locally closed irreducible subset $W_{d,k}$ of $H^S_{d,g}$ of dimension $2d+g+12-k$, where $0 < k \leq d/2$ and $g = (k-1)(2d-3k-2)/2$ is the genus of C.

(2) *If $3 \leq k \leq (d-3)/2$, then the dimension of the Zariski tangent space of $H^S_{d,g}$ at the point $[C]$ is equal to $2d+g+14-k$.*

(3) *The closure $\overline{W}_{d,k}$ of $W_{d,k}$ in $H^S_{d,g}$ becomes a generically non-reduced irreducible component of $H^S_{d,g}$ if*

$$\frac{2d+1-\sqrt{d^2-20d+1}}{6} < k < \min\left\{\frac{2d+1+\sqrt{d^2-20d+1}}{6}, d-5, \frac{d-2}{2}\right\}.$$

Recently it has been proved in [10, Theorem 2.12] that every smooth connected curve on a non-normal cubic cone is a specialization of curves on a non-normal cubic surface which is not a cone.

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Department of Information Environment, Tokyo Denki University, Muzaigakuendai 2-1200, Inzai-shi Chiba, 270-1382, Japan

E-mail address: nasu@sie.dendai.ac.jp